Exercise 3

Solve the differential equation.

$$y'' + 3y = 0$$

Solution

This is a linear homogeneous ODE with constant coefficients, so it has solutions of the form $y = e^{rx}$.

$$y = e^{rx} \quad \rightarrow \quad y' = re^{rx} \quad \rightarrow \quad y'' = r^2 e^{rx}$$

Substitute these formulas into the ODE.

$$r^2 e^{rx} + 3(e^{rx}) = 0$$

Divide both sides by e^{rx}.

$$r^2 + 3 = 0$$

Solve for r.

$$r = \left\{-i\sqrt{3}, i\sqrt{3}\right\}$$

Two solutions to the ODE are $e^{-i\sqrt{3}x}$ and $e^{i\sqrt{3}x}$. According to the principle of superposition, the general solution to the ODE is a linear combination of these two.

$$\begin{aligned} y(x) &= C_1 e^{-i\sqrt{3}x} + C_2 e^{i\sqrt{3}x} \\ &= C_1(\cos\sqrt{3}x - i\sin\sqrt{3}x) + C_2(\cos\sqrt{3}x + i\sin\sqrt{3}x) \\ &= (C_1 + C_2)\cos\sqrt{3}x + (-iC_1 + iC_2)\sin\sqrt{3}x \end{aligned}$$

Therefore,

$$y(x) = C_3\cos\sqrt{3}x + C_4\sin\sqrt{3}x,$$

where C_3 and C_4 are arbitrary constants.